SEMIGROUPS GENERATED BY ELLIPTIC OPERATORS IN NON-DIVERGENCE FORM ON $C_0(\Omega)$

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ABSTRACT. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set satisfying the uniform exterior cone condition. Let $\mathcal{A}$ be a uniformly elliptic operator given by

$$\mathcal{A}u = \sum_{i,j=1}^{n} a_{ij}\partial_{ij}u + \sum_{j=1}^{n} b_j\partial_j u + cu$$

where

$$a_{ij} \in C(\bar{\Omega}) \quad \text{and} \quad b_j, c \in L^\infty(\Omega), c \leq 0 \ .$$

We show that the realization A_0 of $\mathcal{A}$ in

$$C_0(\Omega) := \{u \in C(\bar{\Omega}) : u_{|\partial\Omega} = 0\}$$

given by

$$\begin{aligned} D(A_0) &:= \{u \in C_0(\Omega) \cap W^{2,n}_{\text{loc}}(\Omega) : \mathcal{A}u \in C_0(\Omega)\} \\ A_0 u &:= \mathcal{A}u \end{aligned}$$

generates a bounded holomorphic C_0-semigroup on $C_0(\Omega)$. The result is in particular true if Ω is a Lipschitz domain. So far the best known result seems to be the case where Ω has C^2-boundary [Lun95, Section 3.1.5]. We also study the elliptic problem

$$\begin{aligned} -\mathcal{A}u &= f \\ u_{|\partial\Omega} &= g \ . \end{aligned}$$

0. Introduction

The aim of this paper is to study elliptic and parabolic problems for operators in non-divergence form with continuous second order coefficients and to prove the existence (and uniqueness) of solutions which are continuous up to the boundary of the domain. Throughout this paper Ω is a bounded open set in $\mathbb{R}^n, n \geq 2$, with boundary $\partial\Omega$. We consider the operator $\mathcal{A}$ given by

$$\mathcal{A}u := \sum_{i,j=1}^{n} a_{ij}\partial_{ij}u + \sum_{j=1}^{n} b_j\partial_j u + cu$$

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with real-valued coefficients a_{ij}, b_j, c satisfying

$$\begin{aligned}&b_j \in L^\infty(\Omega)\ ,\ j = 1, \ldots, n\ ,\ c \in L^\infty(\Omega)\ ,\ c \le 0\\ &a_{ij} \in C(\bar\Omega)\ ,\ a_{ij} = a_{ji}\ ,\\ &\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \ge \Lambda|\xi|^2 \qquad (x \in \bar\Omega, \xi \in \mathbb{R}^n)\end{aligned}$$

where $\Lambda > 0$ is a fixed constant.

Our best results are obtained under the hypothesis that Ω satisfies the uniform exterior cone condition (and thus in particular if Ω has Lipschitz boundary). Then we show that for each $f \in L^n(\Omega), g \in C(\partial\Omega)$ there exists a unique $u \in C(\bar\Omega) \cap W^{2,n}_{\text{loc}}(\Omega)$ such that

$$(E)\begin{cases} -\mathcal{A}u &= f\\ u_{|\partial\Omega} &= g\ .\end{cases}$$

(Corollary 2.3). This result is proved with the help of Alexandrov's maximum principle (which is responsible for the choice of $p = n$) and other standard results for elliptic second order differential operators (put together in the appendix). Our main concern is the parabolic problem

$$(P)\begin{cases} u_t &= \mathcal{A}u\\ u(0,\cdot) &= u_0\\ u(t,x) &= 0 \qquad x \in \partial\Omega\ ,\ t > 0\ .\end{cases}$$

with Dirichlet boundary conditions. Let $C_0(\Omega) := \{v \in C(\bar\Omega) : v_{|\partial\Omega} = 0\}$. Under the uniform exterior cone condition, we show that the realization A_0 of $\mathcal{A}$ in $C_0(\Omega)$ given by

$$\begin{aligned} D(A_0) &:= \{v \in C_0(\Omega) \cap W^{2,n}_{\text{loc}}(\Omega) : \mathcal{A}v \in C_0(\Omega)\}\\ A_0 v &:= \mathcal{A}v\end{aligned}$$

generates a bounded, holomorphic C_0-semigroup on $C_0(\Omega)$. This improves the known results, which are presented in the monographie of Lunardi [Lun95, Corollary 3.1.21] for Ω of class C^2 (and b_j, c uniformly continuous).

If the second order coefficients are Lipschitz continuous, then the results mentioned so far hold if Ω is merely Wiener-regular. For elliptic operators in divergence form, this is proved in [GT98, Theorem 8.31] for the elliptic problem (E) and in [AB99, Corollary 4.7] for the parabolic problem (P). Concerning the elliptic problem (E), and in particular the Dirichlet problem; i.e., the case $f = 0$ in (E), there is earlier work by Krylov [Kry67, Theorem 4], who shows well-posedness of the Dirichlet problem if Ω is merely Wiener regular and the second order coefficients are Dini-continuous. Krylov also obtains the well-posedness of the Dirichlet problem for $a_{ij} \in C(\bar\Omega)$ if Ω satisfies the uniform exterior cone condition [Kry67, Theorem 5]. He uses different (partially probabilistic) methods, though.

1. The Poisson problem

We consider the bounded open set $\Omega \subset \mathbb{R}^n$ and the elliptic operator $\mathcal{A}$ from the Introduction. At first we consider the case where the second order conditions are

Lipschitz continuous. Then we merely need a very mild regularity condition on Ω. We say that Ω is *Wiener regular* (or *Dirichlet regular*) if for each $g \in C(\partial\Omega)$ there exists a solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ of the Dirichlet problem

$$\begin{aligned} \Delta u &= 0 \\ u_{|\partial\Omega} &= g \ . \end{aligned}$$

If Ω satisfies the exterior cone condition, then Ω is Dirichlet regular.

Theorem 1.1. *Assume that the second order coefficients a_{ij} are globally Lipschitz continuous. If Ω is Wiener-regular, then for each $f \in L^n(\Omega)$, there exists a unique $u \in W^{2,n}_{\mathrm{loc}}(\Omega) \cap C_0(\Omega)$ such that*

$$-\mathcal{A}u = f \ .$$

The point is that for Lipschitz continuous a_{ij} the operator $\mathcal{A}$ may be written in divergence form. This is due to the following lemma.

Lemma 1.2. *Let $h : \Omega \to \mathbb{R}$ be Lipschitz continuous. Then $h \in W^{1,\infty}(\Omega)$. In particular, $hu \in W^{1,2}(\Omega)$ for all $u \in W^{1,2}(\Omega)$ and $\partial_j(hu) = (\partial_j h)u + h\partial_j u$.*

Proof. One can extend h to a Lipschitz function on $\mathbb{R}^n$ (without increasing the Lipschitz constant, see [Min70]). Now the result follows from [Eva98, 5.8 Theorem 4]. □

Proof of Theorem 1.1. We assume that Ω is Dirichlet regular. Uniqueness follows from Aleksandrov's maximum principle Theorem A.1. In order to solve the problem we replace $\mathcal{A}$ by an operator in divergence form in the following way. Let $\tilde{b}_j := b_j - \sum_{i=1}^{n} \partial_i a_{ij}, j = 1, \ldots, n$. Then $\tilde{b}_j \in L^\infty(\Omega)$. Consider the elliptic operator $\mathcal{A}_d$ in divergence form given by

$$\mathcal{A}_d u = \sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j u) + \sum_{j=1}^{n} \tilde{b}_j \partial_j u + cu \ .$$

a) Let $f \in L^q(\Omega)$ for $q > n$. By [GT98, Theorem 8.31] or [AB99, Corollary 4.6] there exists a unique $u \in C_0(\Omega) \cap W^{1,2}_{\mathrm{loc}}(\Omega)$ such that $-\mathcal{A}_d u = f$ weakly, i.e.,

$$\sum_{i,j=1}^{d} \int_\Omega a_{ij}\partial_j u \partial_i v - \sum_{j=1}^{d} \int_\Omega \tilde{b}_j \partial_j u v - \int_\Omega cuv = \int_\Omega fv$$

for all $v \in \mathcal{D}(\Omega)$ (the space of all test functions). We mention in passing that $u \in W^{1,2}_0(\Omega)$ by [AB99, Lemma 4.2]. For our purposes, it is important that $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ by Friedrich's theorem [GT98, Theorem 8.8]. Here we use again that the a_{ij} are uniformly Lipschitz continuous but do not need any further hypothesis on b_j and c. It follows from Lemma 1.2 that $a_{ij}\partial_j u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and $\partial_i(a_{ij}\partial_j u) = (\partial_i a_{ij})\partial_j u + a_{ij}\partial_{ij} u$. Thus $\mathcal{A}_d u = \mathcal{A}u$. Now it follows from the interior Calderon-Zygmund estimate Theorem A.2 that $u \in W^{2,q}_{\mathrm{loc}}(\Omega) \subset W^{2,n}_{\mathrm{loc}}(\Omega)$. This settles the result if $f \in L^q(\Omega)$ for some $q > n$.

b) Let $f \in L^n(\Omega)$. Choose $f_k \in L^\infty(\Omega)$ sucht that $\lim_{k\to\infty} f_k = f$ in $L^n(\Omega)$. Let

$u_k \in W^{2,n}_{\text{loc}} \cap C_0(\Omega)$ such that $-\mathcal{A}u_k = f_k$ (use case a)). By Aleksandrov's maximum principle Thereom A.1, we have

$$\|u_k - u_\ell\|_{L^\infty(\Omega)} \le c\|f_k - f_\ell\|_{L^n(\Omega)} .$$

Thus u_k converge uniformly to a function $u \in C_0(\Omega)$ as $k \to \infty$. By the Calderon-Zygmund estimate (Theorem A.2),

$$\|u_k\|_{W^{2,n}(B_\varrho)} \le c(\|u_k\|_{L^n(B_{2\varrho})} + \|f_k\|_{L^n(B_{2\varrho})})$$

if $\overline{B_{2\varrho}} \subset \Omega$, where the constant c does not depend on k. Thus the sequence $(u_k)_{k\in\mathbb{N}}$ is bounded in $W^{2,n}(B_\varrho)$. It follows from reflexivity that $u \in W^{2,n}(B_\varrho)$ and $u_k \rightharpoonup u$ in $W^{2,n}(B_\varrho)$ as $k \to \infty$ after extraction of a subsequence. Consequently, $u \in W^{2,n}_{\text{loc}}(\Omega) \cap C_0(\Omega)$. Since $-\mathcal{A}u_k = f_k$ for all $k \in \mathbb{N}$, it follows that $-\mathcal{A}u = f$. □

Now we return to the general assumption $a_{ij} \in C(\bar{\Omega})$ and do no longer assume that the a_{ij} are Lipschitz continuous. We need the following lemma which we prove for convenience.

Lemma 1.3. *a) There exist $\tilde{a}_{ij} \in C^b(\mathbb{R}^n)$ such that $\tilde{a}_{ij} = \tilde{a}_{ji}, \tilde{a}_{ij}(x) = a_{ij}(x)$ if $x \in \Omega$ and*

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \ge \frac{\Lambda}{2}|\xi|^2$$

for all $\xi \in \mathbb{R}^n, x \in \Omega$.
b) There exist $a^k_{ij} \in C^\infty(\bar{\Omega})$ such that $a^k_{ij} = a^k_{ji}$, $\sum\limits_{i,j=1}^n a^k_{ij}(x)\xi_i\xi_j \ge \frac{\Lambda}{2}|\xi|^2$ and $\lim\limits_{k\to\infty} a^k_{ij}(x) = a_{ij}(x)$ uniformly on $\bar{\Omega}$.

Proof. a) Let $b_{ij} : \mathbb{R}^n \to \mathbb{R}$ be a bounded, continuous extension of a_{ij} to $\mathbb{R}^n$. Replacing b_{ij} by $\frac{b_{ij}+b_{ji}}{2}$, we may assume that $b_{ij} = b_{ji}$. Since the function $\varphi : \mathbb{R}^n \times S^1 \to \mathbb{R}$ given by $\varphi(x,\xi) := \sum\limits_{i,j=1}^n b_{ij}(x)\xi_i\xi_j$ is continuous and $S^1 := \{\xi \in \mathbb{R}^n : |\xi| = 1\}$ is compact, the set $\Omega_1 := \{x \in \mathbb{R}^n : \varphi(x,\xi) > \frac{\Lambda}{2}$ for all $\xi \in S^1\}$ is open and contains $\bar{\Omega}$. Let $0 \le \varphi_1, \varphi_2 \in C(\mathbb{R}^n)$ such that $\varphi_1(x) + \varphi_2(x) = 1$ for all $x \in \mathbb{R}^n$ and $\varphi_2(x) = 1$ for $x \in \mathbb{R}^n \setminus \Omega_1, \varphi_1(x) = 1$ for $x \in \bar{\Omega}$. Then $\tilde{a}_{ij} := \varphi_1 b_{ij} + \frac{\Lambda}{2}\varphi_2\delta_{ij}$ fulfills the requirements.
b) Let $(\varrho_k)_{k\in\mathbb{N}}$ be a mollifier satisfying $\operatorname{supp}\varrho_k \subset B_{1/k}(0)$. Then $a^k_{ij} = \tilde{a}_{ij} * \varrho_k \in C^\infty(\mathbb{R}^n)$ and $\lim\limits_{k\to\infty} a^k_{ij}(x) = \tilde{a}_{ij}(x) = a_{ij}(x)$ uniformly in $x \in \bar{\Omega}$. If $\frac{1}{k} < \operatorname{dist}(\partial\Omega_1, \Omega)$, then for $x \in \Omega, \xi \in \mathbb{R}^n$

$$\begin{aligned}
\sum_{i,j=1}^n a^k_{ij}(x)\xi_i\xi_j &= \int\limits_{|y|<1/k} \sum_{i,j=1}^n \tilde{a}_{ij}(x-y)\xi_i\xi_j\,\varrho_k(y)\,dy \\
&\ge \frac{\Lambda}{2} \int\limits_{|y|<1/k} \varrho_k(y)\,dy = \frac{\Lambda}{2} .
\end{aligned}$$

□

Theorem 1.4. *Assume that Ω satisfies the uniform exterior cone condition. Then for all $f \in L^n(\Omega)$ there exists a unique $u \in C_0(\Omega) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$ such that $-\mathcal{A}u = f$.*

Proof. As for Theorem 1.1 we merely have to prove existence of a solution. We choose $a^k_{ij} \in C^\infty(\bar{\Omega})$ as im Lemma 1.3. Let $\mathcal{A}_k$ be the elliptic operator with the second order coefficients a_{ij} of $\mathcal{A}$ replaced by a^k_{ij}. Let $f \in L^n(\Omega)$. By Theorem 1.1, for each $k \in \mathbb{N}$ there exists a unique $u_k \in W^{2,n}_{\mathrm{loc}}(\Omega) \cap C_0(\Omega)$ such that $-\mathcal{A}_k u_k = f$. By Hölder regularity (Theorem A.3) there exists a constant c which does not depend on $k \in \mathbb{N}$ such that

$$\|u_k\|_{C^\alpha(\Omega)} \le c(\|f\|_{L^n(\Omega)} + \|u_k\|_{L^n(\Omega)}) \ .$$

By Aleksandrov's maximum principle $\|u_k\|_{L^\infty(\Omega)} \le 2c_1\|f\|_{L^n(\Omega)}$ for all $k \in \mathbb{N}$ and some constant c_1. Notice that the first order coefficients of $\mathcal{A}_k$ are independent of $k \in \mathbb{N}$. Thus $(u_k)_{k\in\mathbb{N}}$ is bounded in $C^\alpha(\Omega)$. By the Arcela-Ascoli theorem we may assume that u_k converges uniformly to $u \in C_0(\Omega)$ as $k \to \infty$ (passing to a subsequence of necessary). Let $\overline{B_{2\varrho}} \subset \Omega$ where $B_{2\varrho}$ is a ball of radius 2ϱ. Since the modulus of continuity of the a^k_{ij} is bounded, by the interior Calderon-Zygmund estimate Theorem A.2

$$\|u_k\|_{W^{2,n}(B_\varrho)} \le c_2(\|u_k\|_{L^n(B_{2\varrho})} + \|f\|_{L^n(B_{2\varrho})})$$

for all $k \in \mathbb{N}$ and some constant c_2. It follows from reflexivity that $u \in W^{2,n}(B_\varrho)$ and $u_k \rightharpoonup u$ in $W^{2,n}(B_\varrho)$ as $k \to \infty$ after extraction of a subsequence. Since $-\mathcal{A}_k u_k = f$, it follows that $-\mathcal{A}u = f$. In fact, since $u_k \rightharpoonup u$ weakly in $W^{2,n}(B_\varrho)$, it follows that $\partial_{ij}u_k \rightharpoonup \partial_{ij}u$ in $L^n(B_\varrho)$ as $k \to \infty$. Thus $\sup_k \|\partial_{ij}u_k\|_{L^n(B_\varrho)} < \infty$. It follows that

$$(a^k_{ij} - a_{ij})\partial_{ij}u_k \to 0 \ \text{ in } \ L^n(B_\varrho) \ \text{ as } \ k \to \infty$$

and consequently $a^k_{ij}\partial_{ij}u_k \rightharpoonup a_{ij}\partial_{ij}u$ in $L^n(B_\varrho)$. □

2. The Dirichlet problem

In this section we show the equivalence between well-posedness of the *Poisson problem*

$$(P) \qquad \begin{aligned} -\mathcal{A}u &= f \\ u_{|\partial\Omega} &= 0 \end{aligned}$$

and the *Dirichlet problem*

$$(D) \qquad \begin{aligned} \mathcal{A}u &= 0 \\ u_{|\partial\Omega} &= g \end{aligned}$$

where $f \in L^n(\Omega)$ and $g \in C(\partial\Omega)$ are given. We consider the operator $\mathcal{A}$ defined in the previous section and define its realization A in $L^n(\Omega)$ (recall that $\Omega \subset \mathbb{R}^n$) by

$$\begin{aligned} D(A) &:= \{u \in C_0(\Omega) \cap W^{2,n}_{\mathrm{loc}}(\Omega) : \mathcal{A}u \in L^n(\Omega)\} \\ Au &:= \mathcal{A}u \ . \end{aligned}$$

Thus the Poisson problem can be formulated in a more precise way by asking under which conditions A is invertible (i.e. bijective from $D(A)$ to $L^n(\Omega)$ with bounded inverse $A^{-1} : L^n(\Omega) \to L^n(\Omega)$). Note that for $\mu > 0$, the operator $A - \mu := A - \mu I$ has the same form as A (the order-0-coefficient c being just replaced by $c - \mu$).

Proposition 2.1. *The operator A is closed and injective. Thus, A is invertible whenever it is surjective. If $A - \mu$ is invertible for some $\mu \geq 0$, then it is so for all.*

Proof. By the Aleksandrov maximum principle (Theorem A.1) there exists a constant $c_1 > 0$ such that

$$\|u\|_\infty \leq 2c_1 \|\mu u - Au\|_{L^n(\Omega)} \tag{2.1}$$

for all $u \in D(A), \mu \geq 0$. In order to show that A is closed, let $u_k \in D(A)$ such that $u_k \to u$ in $L^n(\Omega)$ and $Au_k \to f$ in $L^n(\Omega)$. It follows from (2.1) that $u \in C_0(\Omega)$ and $\lim_{k\to\infty} u_k = u$ in $C_0(\Omega)$. Let $B_{2\varrho}$ be a ball of radius 2ϱ such that $\overline{B_{2\varrho}} \subset \Omega$. By the Calderon-Zygmund estimate (Theorem A.2)

$$\|u_k\|_{W^{2,n}(B_\varrho)} \leq c_\varrho (\|u_k\|_{L^n(B_{2\varrho})} + \|Au_k\|_{L^n(B_{2\varrho})}) \ .$$

It follows that $(u_k)_{k\in\mathbb{N}}$ is bounded in $W^{2,n}(B_\varrho)$. By passing to a subsequence we can assume that $u_k \rightharpoonup u$ in $W^{2,n}(B_\varrho)$. Consequently $\mathcal{A}u_k \rightharpoonup \mathcal{A}u$ in $L^n(B_\varrho)$. Thus $\mathcal{A}u = f$ on B_ϱ. Since the ball is arbitrary, it follows that $u \in D(A)$ and $Au = f$.

Now assume that $\mu_1 - A$ is invertible for some $\mu_1 \geq 0$. Let $\mu_2 \geq 0$. Define $B(t) = t(\mu_1 - A) + (1 - t)(\mu_2 - A)$. Since $(\mu_1 - A), (\mu_2 - A) \in \mathcal{L}(D(A), L^n(\Omega))$ where $D(A)$ is considered as a Banach space with respect to the graph norm $\|u\|_A := \|u\|_{L^n(\Omega)} + \|Au\|_{L^n(\Omega)}$, since by (2.1)

$$2c_1 \|B(t)u\|_{L^n(\Omega)} \geq \|u\|_{C(\bar\Omega)} \geq \frac{1}{|\Omega|^{1/n}} \|u\|_{L^n(\Omega)} \ ,$$

for all $t \in [0, 1]$ and since $B(1)$ is invertible, it follows from [GT98, Theorem 5.2] that $B(0)$ is also invertible. $\square$

We call a function u on Ω $\mathcal{A}$*-harmonic* if $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ for some $p > 1$ and $\mathcal{A}u = 0$. By [GT98, Theorem 9.16] each $\mathcal{A}$-harmonic function u is in $\bigcap_{q>1} W^{2,q}_{\mathrm{loc}}(\Omega)$. Given $g \in C(\partial\Omega)$, the *Dirichlet problem* consists in finding an $\mathcal{A}$-harmonic function $u \in C(\bar\Omega)$ such that $u_{|\partial\Omega} = g$. We say that Ω is $\mathcal{A}$*-regular* if for each $g \in C(\partial\Omega)$ there is a solution of the Dirichlet problem. Uniqueness follows from the maximum principle [GT98, Theorem 9.6]

$$-\|u^-_{|\partial\Omega}\|_{L^\infty(\partial\Omega)} \leq u(x) \leq \|u^+_{|\partial\Omega}\|_{L^\infty(\partial\Omega)} \tag{2.2}$$

for all $x \in \bar\Omega$, which holds for each $\mathcal{A}$-harmonic function $u \in C(\bar\Omega)$. In particular,

$$\|u\|_{C(\bar\Omega)} \leq \|u\|_{C(\partial\Omega)} \ . \tag{2.3}$$

Theorem 2.2. *The operator A is invertible if and only if Ω is $\mathcal{A}$-regular.*

Proof. a) Assume that A is invertible.
First step: Let $g \in C(\partial\Omega)$ be of the form $g = G_{|\partial\Omega}$ where $G \in C^2(\bar{\Omega})$. Then $\mathcal{A}G \in L^n(\Omega)$. Let $v = A^{-1}(\mathcal{A}G)$, then $u := G - v$ solves the Dirichlet problem for g.
Second step: Let $g \in C(\partial\Omega)$ be arbitrary. Extending g continuously and mollifying we find $g_k \in C(\partial\Omega)$ of the kind considered in the first step such that $g = \lim_{k\to\infty} g_k$ in $C(\partial\Omega)$. Let $u_k \in C(\bar{\Omega})$ be $\mathcal{A}$-harmonic satisfying $u_k|_{\partial\Omega} = g_k$. By (2.3) $u := \lim_{k\to\infty} u_k$ exists in $C(\bar{\Omega})$. In particular, $u|_{\partial\Omega} = g$. Let $\overline{B_{2\varrho}} \subset \Omega$. Then by the Calderon-Zygmund estimate Theorem A.2

$$\|u_k\|_{W^{2,p}(B_\varrho)} \leq c_\varrho \|u_k\|_{L^p(B_{2\varrho})} \leq c_\varrho c \|u_k\|_{C(\bar{\Omega})}$$

(remember that $\mathcal{A}u_k = 0$). Thus $(u_k)_{k\in\mathbb{N}}$ is bounded in $W^{2,p}(B_\varrho)$. Passing to a subsequence, we can assume that $u_k \rightharpoonup u$ in $W^{2,p}(B_\varrho)$. This implies that $\mathcal{A}u = 0$ in B_ϱ. Since the ball is arbitrary, it follows that u is $\mathcal{A}$-harmonic. Thus u is a solution of the Dirichlet problem (D).
b) Conversely, assume that Ω is $\mathcal{A}$-regular. Let $f \in L^n(\Omega)$. We want to find $u \in D(A)$ such that $Au = f$. Let B be a ball containig $\bar{\Omega}$ and extend f by 0 to B. Then by Theorem 1.4 we find $v \in C_0(B) \cap W^{2,n}_{\text{loc}}(B)$ such that $\tilde{\mathcal{A}}v = f$. Here $\tilde{\mathcal{A}}$ is an extension of $\mathcal{A}$ to the ball B according to Lemma 1.3a. Let $g = v_{|\partial\Omega}$. Then by our assumption there exists an $\mathcal{A}$-harmonic function $w \in C(\bar{\Omega})$ such that $w_{|\partial\Omega} = g$. Let $u = v - w$. Then $u \in C_0(\Omega) \cap W^{2,n}_{\text{loc}}(\Omega)$ and $\mathcal{A}u = \mathcal{A}v = f$; i.e. $u \in D(A)$ and $Au = f$. We have shown that A is surjective, which implies invertibility by Proposition 2.1. □

Corollary 2.3. *Assume that one of the following two conditions is satisfied:*
a) Ω is Wiener regular and the coefficients a_{ij} are globally Lipschitz continuous, or
b) Ω satisfies the exterior cone condition.
Then Ω is $\mathcal{A}$-regular. More generally, for all $f \in L^n(\Omega), g \in C(\partial\Omega)$ there exists a unique $u \in C(\bar{\Omega}) \cap W^{2,n}_{\text{loc}}(\Omega)$ satisfying

$$\begin{aligned} -\mathcal{A}u &= f \\ u_{|\partial\Omega} &= g\ . \end{aligned}$$

Proof. Since A is closed by Proposition 2.1 it follows from Theorem 1.1 (in the case a)) and from Theorem 1.4 (in the case b)) that A is invertible. Thus Ω is $\mathcal{A}$ regular by Theorem 2.2. Let $f \in L^n(\Omega), g \in C(\partial\Omega)$. Since Ω is $\mathcal{A}$-regular, there exists an $\mathcal{A}$-harmonic function $u_1 \in C(\bar{\Omega})$ such that $u_{1_{|\partial\Omega}} = g$. Since A is invertible, there exists a function $u_0 \in C_0(\Omega) \cap W^{2,n}_{\text{loc}}(\Omega)$ such that $-\mathcal{A}u_0 = f$. Let $u := u_0 + u_1$. Then $u \in C(\bar{\Omega}) \cap W^{2,n}_{\text{loc}}(\Omega), u_{|\partial\Omega} = g$ and $-\mathcal{A}u = f$. Uniqueness follows from Theorem A.1. □

For the Laplacian $\mathcal{A} = \Delta$, Δ-regularity is the usual regularity of Ω with respect to the classical Dirichlet problem, which is frequently called *Wiener-regularity* because of Wiener's characterization via capacity [GT98, (2.37)]. It is a most interesting question how $\mathcal{A}$-regularity and Δ-regularity are related. In general it is not true

that $\mathcal{A}$-regularity implies Wiener regularity. In fact, K. Miller [Mil70] gives an example of an elliptic operator $\mathcal{A}$ with $b_j = c = 0$ such that the pointed unit disc $\{x \in \mathbb{R}^2 : 0 < |x| < 1\}$ is $\mathcal{A}$-regular even though it is not Δ-regular. The other implication seems to be open. The fact that the uniform exterior cone property (which is much stronger than Δ-regularity) implies $\mathcal{A}$-regularity (Corollary 2.3) had been proved before by Krylov [Kry67, Theorem 5] with the help of probabilistic methods. If Ω is merely Δ-regular, then it seems not to be known whether Ω is $\mathcal{A}$-regular. Known results concerning this question are based on further restrictive conditions on the coefficients a_{ij}. In Theorem 1.1 we gave a proof for globally Lipschitz continuous a_{ij}. The best result seems to be [Kry67, Theorem 4] which goes in both directions: If the a_{ij} are Dini-continuous (in particular, if they are Hölder-continuous), then Ω is Δ-regular if and only if Ω is $\mathcal{A}$-regular.

3. Generation results

An operator B on a complex Banach space X is said to *generate a bounded holomorphic semigroup* if $(\lambda - B)$ is invertible for $\operatorname{Re}\lambda > 0$ and

$$\sup_{\operatorname{Re}\lambda>0} \|\lambda(\lambda - B)^{-1}\| < \infty \ .$$

Then there exist $\theta \in (0, \pi/2)$ and a holomorphic bounded function $T : \Sigma_\theta \to \mathcal{L}(X)$ satisfying $T(z_1 + z_2) = T(z_1)T(z_2)$ such that

$$\lim_{n\to\infty} e^{tB_n} = T(t) \ \text{ in } \ \mathcal{L}(X) \tag{3.1}$$

for all $t > 0$, where $B_n = nB(n-B)^{-1} \in \mathcal{L}(X)$. Here Σ_θ is the sector $\Sigma_\theta := \{re^{i\alpha} : r > 0, |\alpha| < \theta\}$.

If B is an operator on a reel Banach space X we say that B *generates a bounded holomorphic semigroup* if its linear extension $B_\mathbb{C}$ to the complexification $X_\mathbb{C}$ of X generates a bounded holomorphic semigroup $T_\mathbb{C}$ on $X_\mathbb{C}$. In that case $T_\mathbb{C}(t)X \subseteq X$ (see [Lun95, Corollary 2.1.3]); in particular $T(t) := T_\mathbb{C}(t)_{|X} \in \mathcal{L}(X)$. We call $T = (T(t))_{t>0}$ the semigroup generated by B. It satisfies $\lim_{t\downarrow 0} T(t)x = x$ for all $x \in X$ (i.e., it is a C_0-semigroup) if and only if $\overline{D(B)} = X$. We refer to [Lun95, Chapter 2] and [ABHN01, Sec. 3.7] for these facts and further information.

In this section we consider the parts A_c and A_0 of $\mathcal{A}$ in $C(\bar{\Omega})$ and $C_0(\Omega)$ as follows:

$$\begin{aligned} D(A_c) &:= \{u \in C_0(\Omega) \cap W^{2,n}_{\text{loc}}(\Omega) : \mathcal{A}u \in C(\bar{\Omega})\} \\ A_c u &:= \mathcal{A}u \quad \text{and} \\ D(A_0) &:= \{u \in C_0(\Omega) \cap W^{2,n}_{\text{loc}}(\Omega) : \mathcal{A}u \in C_0(\Omega)\} \\ A_0 u &:= \mathcal{A}u \ . \end{aligned}$$

Thus A_c is the part of A in $C(\bar{\Omega})$ and A_0 the part of A_c in $C_0(\Omega)$. Note that $D(A_0) \subseteq D(A_c) \subseteq \bigcap_{q>1} W^{2,q}_{\text{loc}}$ by [GT98, Lemma 9.16]. The main result of this section is the following.

Theorem 3.1. *Assume that Ω is $\mathcal{A}$-regular. Then A_c generates a bounded holomorphic semigroup T on $C(\bar{\Omega})$. The operator A_0 generates a bounded holomorphic C_0-semigroup T_0 on $C_0(\Omega)$. Moreover, $T(t)C_0(\Omega) \subseteq C_0(\Omega)$ and*

$$T_0(t) = T(t)_{|C_0(\Omega)} \ .$$

Recall that Ω is $\mathcal{A}$-regular if one of the following conditions is satisfied:

(a) Ω satisfies the uniform exterior cone condition or
(b) Ω is Wiener regular and the coefficients a_{ij} are Dini-continuous.
In particular, Ω is $\mathcal{A}$-regular if
(a') Ω is a Lipschitz-domain or
(b') Ω is Wiener-regular and the a_{ij} are Hölder continuous.

In the following complex maximum principle (Proposition 3.3) we extend $\mathcal{A}$ to the complex space $W^{2,p}_{\text{loc}}(\Omega)$ without changing the notation. We first proof a lemma.

Lemma 3.2. *Let $B \subseteq \Omega$ be a ball of center x_0 and let $u \in W^{2,p}(B), p > n$, be a complex-valued function such that $\mathcal{A}u \in C(B)$. If $|u(x_0)| \geq |u(x)|$ for all $x \in B$, then*

$$\text{Re}\left[\overline{u(x_0)}(\mathcal{A}u)(x_0\right] \leq 0 \ .$$

Proof. We may assume that $x_0 = 0$. If the claim is wrong, then there exist $\varepsilon > 0$ and a ball $B_\varrho \subset B$ such that $\text{Re}\left[\overline{u(x)}(\mathcal{A}u)(x)\right] \geq \varepsilon$ on B_ϱ.
Since $\partial_j|u|^2 = (\partial_j u)\bar{u} + u\overline{\partial_j u} = 2\text{Re}\,[\partial_j u\bar{u}]$, and $\partial_{ij}(u\bar{u}) = (\partial_{ij}u)\bar{u} + \partial_i u\overline{\partial_j u} + \partial_j u\overline{\partial_i u} + u\overline{\partial_{ij}u}$, and since by ellipticity

$$\text{Re}\sum_{i,j} a_{ij}\partial_i u\overline{\partial_j u} \geq 0 \ , \ \text{Re}\sum_{i,j} a_{ij}\partial_j u\overline{\partial_i u} \geq 0 \ ,$$

it follows that

$$\begin{aligned} \mathcal{A}|u|^2 &\geq \text{Re}\sum_{i,j} a_{ij}(\partial_{ij}u)\bar{u} + \text{Re}\sum_{i,j} a_{ij}u\overline{\partial_{ij}u} \\ &\quad + \sum_j b_j 2\text{Re}\,[\partial_j u\bar{u}] + cu\bar{u} \\ &\geq 2\text{Re}\,(\mathcal{A}u\bar{u}) \geq 2\varepsilon \ \text{ on } \ B_\varrho \ . \end{aligned}$$

Let $\psi(x) = |u|^2 - \tau|x|^2, \tau > 0$. Then $\mathcal{A}|\psi|^2 \geq 2\varepsilon - c_1\tau$ on B_ϱ for all $\tau > 0$ and some $c_1 > 0$. Choosing $\tau > 0$ small enough, we have $\mathcal{A}|\psi|^2 \geq \varepsilon$ on B_ϱ.
Since $\psi \in W^{2,p}(B_\varrho) \cap C(\overline{B_\varrho})$, by Aleksandrov's maximum principle [GT98, Theorem 9.1], see Theorem A.1, it follows that

$$\begin{aligned} |u(0)|^2 = |\psi(0)|^2 &\leq \sup_{\partial B_\varrho(0)} \psi \\ &= \sup_{\partial B_\varrho(0)} |u|^2 - \tau\varrho^2 \\ &\leq |u(0)|^2 - \tau\varrho^2 < |u(0)|^2 \ , \end{aligned}$$

a contradiction. $\square$

Proposition 3.3. *(complex maximum principle). Let $u \in C(\bar{\Omega}) \cap W^{2,n}_{\text{loc}}(\Omega)$ such that $\lambda u - \mathcal{A}u = 0$ where $\operatorname{Re}\lambda > 0$. If there exists $x_0 \in \Omega$ such that $|u(x)| \le |u(x_0)|$ for all $x \in \Omega$, then $u \equiv 0$. Consequently,*

$$\max_{\bar{\Omega}} |u(x)| = \max_{\partial\Omega} |u(x)| \ .$$

Proof. If $|u(x)| \le |u(x_0)|$ for all $x \in \Omega$, then by Lemma 3.2, $\operatorname{Re}\left[\overline{u(x_0)}(\mathcal{A}u)(x_0)\right] \le 0$. Since $\lambda u = \mathcal{A}u$, it follows that

$$\operatorname{Re}\lambda |u(x_0)|^2 = \operatorname{Re}\left[\overline{u(x_0)}(\mathcal{A}u)(x_0)\right] \le 0 \ .$$

Hence $u(x_0) = 0$. □

Next, recall that an operator B on a real Banach space X is called *m-dissipative* if $\lambda - B$ is invertible and

$$\lambda \|(\lambda - B)^{-1}\| \le 1 \quad \text{for all} \quad \lambda > 0 \ .$$

Now we show that the operator A_c is m-dissipative and that the resolvent is positive (i.e., maps non-negative functions to non-negative functions).

Proposition 3.4. *Assume that Ω is $\mathcal{A}$-regular. Then A_c is m-dissipative and $(\lambda - A_c)^{-1} \ge 0$ for $\lambda > 0$.*

Proof. Let $\lambda > 0$. Since by Theorem 2.2 the operator $(\lambda - A)$ is bijective, also $(\lambda - A_c)$ is bijective.
a) We show that $(\lambda - A_c)^{-1} \ge 0$. Let $f \in C(\bar{\Omega}), f \le 0, u := (\lambda - A_c)^{-1} f$. Assume that $u^+ \ne 0$. Since $u \in C_0(\Omega)$, there exists $x_0 \in \Omega$ such that $u(x_0) = \max_{\Omega} u > 0$. Then by Lemma 3.2, $\mathcal{A}u(x_0) \le 0$. Since $\lambda u - \mathcal{A}u = f$, it follows that $\lambda u(x_0) \le f(x_0) \le 0$ a contradiction.
b) Let $f \in C(\bar{\Omega}), u = (\lambda - A_c)^{-1} f$. We show that $\|\lambda u\|_{C(\bar{\Omega})} \le \|f\|_{C(\bar{\Omega})}$. Assume first that $f \ge 0, f \ne 0$. Then $u \ge 0$ by a) and $u \ne 0$. Let $x_0 \in \Omega$ such that $u(x_0) = \|u\|_{C(\bar{\Omega})}$. Then $(A_c u)(x_0) \le 0$ by Lemma 3.2. Hence $\lambda u(x_0) \le \lambda u(x_0) - (A_c u)(x_0) = f(x_0) \le \|f\|_{C(\bar{\Omega})}$.
If $f \in C(\bar{\Omega})$ is arbitrary, then by a) $|(\lambda - A_c)^{-1} f| \le (\lambda - A_c)^{-1}|f|$ and so $\|\lambda(\lambda - A_c)^{-1} f\|_{C(\bar{\Omega})} \le \|f\|_{C(\bar{\Omega})}$. □

Now we consider the complex extension of A_c (still denoted by A_c) to the space of all complex-valued functions on $\bar{\Omega}$ which we still denote by $C(\bar{\Omega})$. Our aim is to prove that for $\operatorname{Re}\lambda > 0$ the operator $(\lambda - A_c)^{-1}$ is invertible and

$$\|(\lambda - A_c)^{-1}\| \le \frac{M}{|\lambda|} \ ,$$

where M is a constant. For that, we extend the coefficents a_{ij} to uniformly continuous bounded real-valued functions on $\mathbb{R}^n$ satisfying the strict ellipticity condition

$$\operatorname{Re} \sum_{i,j=1}^{d} a_{ij}(x) \xi_i \bar{\xi}_j \ge \frac{\Lambda}{2} |\xi|^2$$

($\xi \in \mathbb{R}^n, x \in \mathbb{R}^n$), keeping the some notation, see Lemma 1.3a. We extend b_j, c to bounded measurable functions on $\mathbb{R}^n$ such that $c \leq 0$ (keeping the same notation). Now we define the operator B_∞ on $L^\infty(\mathbb{R}^n)$ by

$$\begin{array}{lll} D(B_\infty) & := & \{u \in \bigcap\limits_{p>1} W^{2,p}_{\text{loc}}(\mathbb{R}^n) : u, \mathcal{B}u \in L^\infty(\mathbb{R}^n)\} \\ B_\infty u & := & \mathcal{B}u \ , \\ B_\infty u & := & \sum\limits_{i,j=1}^{d} a_{ij}\partial_{ij}u + \sum\limits_{j=1}^{d} b_j\partial_j u + cu \text{ for } u \in W^{2,p}_{\text{loc}}(\mathbb{R}^n) \ . \end{array}$$

where

The operator B_∞ is sectorial. This is proved in [Lun95, Theorem 3.1.7] under the assumption that the coefficients b_j, c are uniformly continuous. We give a perturbation argument to deduce the general case from the case $b_j = c = 0$. The following lemma shows in particular that the domain of B_∞ is independent of b_j and c.

Lemma 3.5. *One has $D(B_\infty) \subset W^{1,\infty}(\mathbb{R}^n)$. Moreover, for each $\varepsilon > 0$ there exists $c_\varepsilon \geq 0$ such that*

$$\|u\|_{W^{1,\infty}(\mathbb{R}^n)} \leq \varepsilon \|B_\infty u\|_{L^\infty(\mathbb{R}^n)} + c_\varepsilon \|u\|_{L^\infty(\mathbb{R}^n)}$$

for all $u \in D(B_\infty)$.

Proof. Consider an arbitrary ball B_1 in $\mathbb{R}^n$ of radius 1 and the corresponding ball B_2 of radius 2. Let $p > n$. Since the injection of $W^{2,p}(B_1)$ into $C^1(\bar{B}_1)$ is compact, for each $\varepsilon > 0$ there exists $c'_\varepsilon > 0$ such that

$$\|u\|_{C^1(\bar{B}_1)} \leq \varepsilon \|u\|_{W^{2,p}(B_1)} + c'_\varepsilon \|u\|_{L^\infty(B_1)} \ .$$

By the Calderon-Zygmund estimate this implies that

$$\begin{array}{lll} \|u\|_{C^1(\bar{B}_1)} & \leq & \varepsilon c_1(\|B_\infty u\|_{L^\infty(B_2)} + \|u\|_{L^\infty(B_2)}) \\ & & +c'_\varepsilon \|u\|_{L^\infty(B_1)} \\ & \leq & \varepsilon c_1 \|B_\infty u\|_{L^\infty(\mathbb{R}^n)} + (\varepsilon c_1 + c'_\varepsilon) \cdot \|u\|_{L^\infty(\mathbb{R}^n)} \ . \end{array}$$

Since $\|u\|_{L^\infty(\mathbb{R}^n)} = \sup\limits_{B_1} \|u\|_{L^\infty(B_1)}$, where the supremum is taken over all balls of radius 1 in $\mathbb{R}^n$, the claim follows. ☐

Theorem 3.6. *There exist $M \geq 0, \omega \in \mathbb{R}$ such that $(\lambda - B_\infty)$ is invertible and*

$$\|\lambda(\lambda - B_\infty)^{-1}\| \leq M \quad (\operatorname{Re} \lambda > \omega) \ .$$

Proof. Denote by B^0_∞ the operator with the coefficients b_j, c replaced by 0. Lemma 3.5 implies that $D(B^0_\infty) = D(B_\infty)$ and (applied to B^0_∞) that

$$\|(B_\infty - B^0_\infty)u\|_{L^\infty(\mathbb{R}^n)} \leq \varepsilon \|B^0_\infty u\|_{L^\infty(\mathbb{R}^n)} + c'_\varepsilon \|u\|_{L^\infty(\mathbb{R}^n)}$$

for all $u \in D(B^0_\infty), \varepsilon > 0$ and some $c'_\varepsilon \geq 0$. Since B^0_∞ is sectorial by [Lun95, Theorem 3.1.7] the claim follows from the usual holomorphic perturbation result [ABHN01, Theorem 3.7.23]. ☐

Now we use the maximum principle, Lemma 3.2, to carry over the sectorial estimate from $\mathbb{R}^n$ to Ω. This is done in a very abstract framework by Lumer-Paquet [LP77], see [Are04, Section 2.5] for the Laplacian.

Proof of Theorem 3.1. Let ω be the constant from Theorem 3.6 and let $\operatorname{Re}\lambda > \omega, f \in C(\bar{\Omega}), u = (\lambda - A_c)^{-1} f$. Then

$$u \in C_0(\Omega) \cap \bigcap_{p>1} W^{2,p}_{\mathrm{loc}}(\Omega) \quad \text{and} \quad \lambda u - \mathcal{A}u = f \ .$$

Extend f by 0 to $\mathbb{R}^n$ and let $v = (\lambda - B_\infty)^{-1} f$. Then $\lambda v - \mathcal{A}v = f$ on Ω and $\|\lambda v\|_{L^\infty(\Omega)} \leq M\|f\|_{C(\bar{\Omega})}$ by Theorem 3.6. Moreover, $w := v - u \in C(\bar{\Omega}) \cap \bigcap_{p\geq 1} W^{2,p}_{\mathrm{loc}}(\Omega), \lambda w - \mathcal{A}w = 0$ on Ω and $w(z) = v(z)$ for all $z \in \partial\Omega$. Then by the complex maximum principle Proposition 3.3,

$$\|w\|_{C(\bar{\Omega})} = \max_{z\in\partial\Omega} |v(z)| \leq \frac{M}{|\lambda|}\|f\|_{C(\bar{\Omega})} \ .$$

Consequently,

$$\begin{aligned} \|u\|_{C(\bar{\Omega})} &= \|u - v + v\|_{C(\bar{\Omega})} \\ &\leq \|w\|_{C(\bar{\Omega})} + \|v\|_{C(\bar{\Omega})} \\ &\leq \frac{2M}{|\lambda|}\|f\|_{C(\bar{\Omega})} \ . \end{aligned}$$

This is the desired estimate which shows that A_c is sectorial. By [Lun95, Proposition 2.1.11] there exist a sector $\Sigma_\theta + \omega := \{\omega + re^{i\alpha} : r > 0, |\alpha| < \theta\}$ with $\theta \in (\frac{\pi}{2}, \pi)$, $\omega \geq 0$, and a constant $M_1 > 0$ such that

$$(\lambda - A_c)^{-1} \ \text{ exists for } \ \lambda \in \Sigma_\theta + \omega \ \text{ and } \ \|\lambda(\lambda - A_c)^{-1}\| \leq M_1 \ .$$

Thus there exists $r > 0$ such that $(\lambda - A_c)$ is invertible and $\|\lambda(\lambda - A_c)^{-1}\| \leq M$ whenever $\operatorname{Re}\lambda > 0$ and $|\lambda| > r$. Since A is invertible by Theorem 2.2, it follows that A_c is bijective. Since the resolvent set of A_c is nonempty, A_c is closed. Thus A_c is invertible. Since by Proposition 3.4 A_c is resolvent positive, it follows from [ABHN01, Proposition 3.11.2] that there exists $\varepsilon > 0$ such that $(\lambda - A_c)$ is invertible whenever $\operatorname{Re}\lambda > -\varepsilon$. As a consequence,

$$\sup_{\substack{|\lambda|\leq r \\ \operatorname{Re}\lambda>0}} \|\lambda(\lambda - A_c)^{-1}\| < \infty \ .$$

Together with the previous estimates this implies that

$$\|\lambda(\lambda - A_c)^{-1}\| \leq M_2$$

whenever $\operatorname{Re}\lambda > 0$ for some constant M_2. Thus A_c generates a bounded holomorphic semigroup T on $C(\bar{\Omega})$. Since $D(A_c) \subset C_0(\Omega)$ and $\mathcal{D}(\Omega) \subset D(A_c)$ it follows that $\overline{D(A_c)} = C_0(\Omega)$. The part of A_c in $C_0(\Omega)$ is A_0. So it follows from [Lun95, Remark 2.1.5, Proposition 2.1.4] that A_0 generates a bounded, holomorphic C_0-semigroup T_0 on $C_0(\Omega)$ and $T_0(t) = T(t)_{|C_0(\Omega)}$ on $C_0(\Omega)$. □

Finally we mention compactness and strict positivity.

Proposition 3.7. *Assume that Ω satisfies the uniform exterior cone condition. Then $(\lambda - A_c)^{-1}$ and $T(t)$ are compact operators $(\lambda > 0, t > 0)$.*

Proof. It follows from Theorem A.3 that $D(A_c) \subset C^\alpha(\Omega)$. Since the embedding of $C^\alpha(\Omega)$ into $C(\bar\Omega)$ is compact, it follows that the resolvent of A_c is compact. Since T is holomorphic, it follows that $T(t)$ is compact for all $t > 0$. □

Proposition 3.8. *Assume that Ω is $\mathcal{A}$-regular. Let $t > 0, 0 \le f \in C_0(\Omega), f \not\equiv 0$. Then $(T_0(t)f)(x) > 0$ for all $x \in \Omega$.*

Proof. a) We show that $u := (\lambda - A_0)^{-1} f$ is strictly positive. Assume that $u(x) \le 0$ for some $x \in \Omega$. Let $v = -u$. Then $\mathcal{A}v - \lambda v = f \ge 0$. It follows from the maximum principle [GT98, Theorem 9.6] that v is constant. Since $v \in C_0(\Omega)$, it follows that $v \equiv 0$. Hence also $f \equiv 0$.
b) It follows from a) that T_0 is a positive, irreducible C_0-semigroup on $C_0(\Omega)$. Since the semigroup is holomorphic, the claim follows from [Na86, C-III.Theorem 3.2.(b)]. □

Appendix A. Results on elliptic partial differential equations

In this section, we collect some results on elliptic partial differential equations, which can be found in text books, for example [GT98]. We consider the elliptic operator $\mathcal{A}$ from the Introduction and assume that the ellipticity constant $\Lambda > 0$ is so small that $\|a_{ij}\|_{L^\infty}$, $\|b_j\|_{L^\infty}$, $\|c\|_{L^\infty} \le \frac{1}{\Lambda}$.

Theorem A.1 (Aleksandrov's maximum principle, [GT98, Theorem 9.1])**.** *Let $f \in L^n(\Omega), u \in C(\bar\Omega) \cap W^{2,n}_{\rm loc}(\Omega)$ such that*

$$-\mathcal{A}u \le f\ .$$

Then

$$\sup_\Omega u \le \sup_{\partial\Omega} u^+ + c_1 \|f^+\|_{L^n(\Omega)}$$

where the constant c_1 depends merely on $n, \operatorname{diam}\Omega$ and $\|b_j\|_{L^n(\Omega)}, j = 1\ldots, n$. Consequently, if $u \in C_0(\Omega)$ and $-\mathcal{A}u = f$, then

$$\|u\|_{L^\infty(\Omega)} \le 2c_1 \|f\|_{L^n(\Omega))}$$

and $u \le 0$ if $f \le 0$.

Theorem A.2 (Interior Calderon-Zygmund estimate, [GT98, Theorem 9.11])**.** *Let $B_{2\varrho}$ be a ball of radius 2ϱ such that $\overline{B_{2\varrho}} \subset \Omega$, and let $u \in W^{2,p}(B_{2\varrho})$, where $1 < p < \infty$. Then*

$$\|u\|_{W^{2,p}(B_\varrho)} \le c_\varrho(\|\mathcal{A}u\|_{L^p(B_{2\varrho})} + \|u\|_{L^p(B_{2\varrho})})$$

where B_ϱ is the ball of radius ϱ concentric with $B_{2\varrho}$. The constant c merely depends on Λ, n, ϱ, p and the continuity moduli of the a_{ij}.

Theorem A.3 (Hölder regularity, [GT98, Corollary 9.29]). *Assume that* Ω *satisfies the uniform exterior cone condition. Let* $u \in C_0(\Omega) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$ *and* $f \in L^n(\Omega)$ *such that* $-\mathcal{A}u = f$. *Then* $u \in C^\alpha(\Omega)$ *and*

$$\|u\|_{C^\alpha(\Omega)} \leq C(\|f\|_{L^n(\Omega)} + \|u\|_{L^2(\Omega)})$$

where $\alpha > 0$ *and* $c > 0$ *depend merely on* Ω, Λ *and* n.

In [GT98, Corollary 9.29] it is supposed that $u \in W^{2,n}(\Omega)$. But an inspection of the proof and of the results preceding [GT98, Corollary 9.29] shows that $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$ suffices. The above Hölder regularity also holds for solutions of equations in divergence form when the right-hand side f is in $L^q(\Omega)$ for some $q > \frac{n}{2}$, see [GT98, Theorem 8.29].

References

[AB99] Arendt, W., Bénilan, Ph.: *Wiener regularity and heat semigroups on spaces of continuous functions.* Progress in Nonlinear Differential Equations and Their Applications **Vol. 35** Birkhäuser Basel 1999, 29–49.

[ABHN01] Arendt, W., Batty, C., Hieber, M., Neubrander, F.: *Vector-valued Laplace Transforms and Cauchy Problems.* Monographs in Mathematics. Birkhäuser, Basel, (2001) ISBN 3-7643-6549-8.

[ADN59] Agmon, S., Douglis, A., Nirenberg, L.: *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I*, Communications on Pure and Applied Mathematics, **12** (1959), pp. 623-727.

[ADN64] Agmon, S., Douglis, A., Nirenberg, L.: *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II*, *Communications on Pure and Applied Mathematics*, **17** (1964), pp. 35-92.

[Are04] Arendt, W. *Semigroups and Evolution Equations: Functional Calculus, Regularity and Kernel Estimates.* Handbook of Differential Equations. Evolutionary Equations, **Vol. 1**. C.M. Dafermos and E. Feireisl eds., Elsevier (2004), 1–85.

[GT98] Gilbarg, D., Trudinger, N.S.: *Elliptic Partial Differential Equations of Second Order*, Springer Verlag, 3.Auflage, Berlin (1998).

[Eva98] Evans, L.C.: *Partial Differential Equations.* American Math. Soc., Providence, R. I. 1998.

[Kry67] Krylov, N. V.: *The first boundary value problem for elliptic equations of second order.* Differencial'nye Uravnenja **3** (1967), 315–326.

[LP77] Lumer, G., Paquet, L.: *Semi-groupes holomorphes et équations d'évolution*, CR Acad. Sc. Paris **284** Série A (1977), pp. 237–240.

[Lun95] Lunardi, A.: *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhäuser Basel, (1995).

[Mil70] Miller, K.: *Nonequivalence of regular boundary points for the Laplace and non-divergence equations, even with continuous coefficients.* Ann. Scuola Norm. Sup. Pisa **3** 24 (1970), 159–163.

[Min70] Minty, G. J.: *On the extension of Lipschitz, Lipschitz-Hölder continuous, and monotone functions.* Bull. Amer. Math. Soc. **76** (1970), 334–339.

[Na86] Nagel, R. (ed.): *One-parameter Semigroups of Positive Operators.* Springer **LN 1184**, (1986) Berlin.

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